

# A More Efficient Simulation Algorithm on Kripke Structures

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## Abstract

A number of algorithms for computing the simulation preorder on Kripke structures are available. The simulation algorithm by Gentilini, Piazza and Policriti (GPP) — subsequently corrected by van Glabbeek and Ploeger — reaches the best space bound, while the simulation algorithm by Ranzato and Tapparo (RT) attains the best time bound. Let  $\Sigma$  denote the state space,  $\rightarrow$  the transition relation and  $P_{\text{sim}}$  the partition of  $\Sigma$  induced by simulation equivalence. We present a novel space and time Efficient Simulation algorithm, called ESim, that runs in  $O(|P_{\text{sim}}||\rightarrow| \log |\Sigma|)$  time and  $O(|P_{\text{sim}}|^2 \log |P_{\text{sim}}| + |\Sigma| \log |\Sigma|)$  space. ESim reaches the best space bound of GPP and significantly improves the GPP time bound by replacing a multiplicative factor  $|P_{\text{sim}}|$  with  $\log |\Sigma|$ . Moreover, ESim significantly improves the space bound of RT by replacing a multiplicative factor  $|\Sigma| \log |\Sigma|$  with  $|P_{\text{sim}}| \log |P_{\text{sim}}|$  while closely approaching the best time bound of RT.

## 1 Introduction

Simulation preorder is a fundamental behavioral relation widely used in process algebra for establishing system correctness and in model checking as a suitable abstraction for reducing state spaces [2, 6]. The problem of efficiently computing the simulation preorder (and equivalence) on finite Kripke structures has been thoroughly investigated and yielded a number of simulation algorithms [3, 4, 5, 7, 9, 11, 12, 15, 21, 22]. Both time and space complexities play important roles in simulation algorithms, since in several applications, especially in model checking, memory requirements may become a serious bottleneck as the input transition system grows.

**State of the Art.** The best simulation algorithms are those by Gentilini, Piazza and Policriti (GPP) [10, 11] — subsequently corrected by van Glabbeek and Ploeger in [12] — and by Ranzato and Tapparo (RT) [20, 21]. Consider a finite Kripke structure where  $\Sigma$  denotes the state space,  $\rightarrow$  the transition relation and  $P_{\text{sim}}$  the partition of  $\Sigma$  induced by simulation equivalence. The algorithm GPP attains a space bound of  $O(|P_{\text{sim}}|^2 \log |P_{\text{sim}}| + |\Sigma| \log |\Sigma|)$  while it runs in  $O(|P_{\text{sim}}|^2 |\rightarrow|)$  time, where these are both bit complexities (i.e., the word size is a single bit). Let us remark that both [11, 12] state the bit space complexity of GPP is in  $O(|P_{\text{sim}}|^2 + |\Sigma| \log |P_{\text{sim}}|)$ . However, this is not correct. In fact, GPP [11, Section 4, p. 98] assumes that the states belonging to some block are stored as a doubly linked list in order to remove and add states in constant time. Hence, this requires that the pointers in these doubly linked lists are able to distinguish  $|\Sigma|$  states, so that its bit space complexity is in  $O(|\Sigma| \log |\Sigma|)$ . Furthermore, GPP uses Henzinger, Henzinger and Kopke [15] algorithm (HKK), which is called on a Kripke structure where states are blocks of the current partition. Since the bit space complexity of HHK is in  $O(|\Sigma|^2 \log |\Sigma|)$ , due to the sets of states  $Rem(s)$  which are stored as lists, this adds a factor  $O(|P_{\text{sim}}|^2 \log |P_{\text{sim}}|)$  to the bit space complexity of GPP, which therefore results to be  $O(|P_{\text{sim}}|^2 \log |P_{\text{sim}}| + |\Sigma| \log |\Sigma|)$ . It is worth remarking that a space complexity in  $O(|P_{\text{sim}}|^2 + |\Sigma| \log |P_{\text{sim}}|)$  can be considered optimal for a simulation algorithm, since this is of the same order as the size of the output, which needs  $O(|P_{\text{sim}}|^2)$  space for storing the simulation preorder as a partial order on simulation equivalence classes and  $O(|\Sigma| \log |P_{\text{sim}}|)$  space for storing the simulation equivalence class for any state. Hence, the bit space complexity of GPP can be considered quasi-optimal. On the other hand, the algorithm RT features the best time bound  $O(|P_{\text{sim}}||\rightarrow|)$  while it takes  $O(|P_{\text{sim}}||\Sigma| \log |\Sigma|)$  space. Let us also mention the simulation algorithm by Crafa, Ranzato

Algorithm	Space complexity	Time complexity
RT [21]	$O( P_{\text{sim}}  \Sigma  \log  \Sigma )$	$O( P_{\text{sim}}  \rightarrow )$
CRT [9]	$O(( P_{\text{sp}}  P_{\text{sim}}  +  \Sigma ) \log  P_{\text{sp}} )$	$O( P_{\text{sim}}  \rightarrow  +  P_{\text{sim}} ^2 \rightarrow^{P_{\text{sp}}, P_{\text{sim}}} )$
GPP [11]	$O( P_{\text{sim}} ^2 \log  P_{\text{sim}}  +  \Sigma  \log  \Sigma )$	$O( P_{\text{sim}} ^2 \rightarrow )$
ESim (this paper)	$O( P_{\text{sim}} ^2 \log  P_{\text{sim}}  +  \Sigma  \log  \Sigma )$	$O( P_{\text{sim}}  \rightarrow  \log  \Sigma )$

Table 1: Space and time complexities of simulation algorithms.

and Tapparo [9] (CRT) which is a kind of compromise between GPP and RT: on the one hand, CRT retains a space complexity comparable with that of GPP while improving its time complexity of GPP-GP; on the other hand, CRT improves the space complexity of RT while significantly worsening its time complexity. The situation for RT, CRT and GPP is summarized in Table 5.1, where for CRT we have that  $P_{\text{sp}}$  is a state partition such that  $|P_{\text{sim}}| \leq |P_{\text{sp}}| \leq |\text{Bisimulation Equivalence}|$  (in practice  $|P_{\text{sim}}| \approx |P_{\text{sp}}|$ , as shown in [9]) and  $\rightarrow^{P_{\text{sp}}, P_{\text{sim}}} = \{(B, C) \mid B \in P_{\text{sp}}, C \in P_{\text{sim}}, B \rightarrow \exists C\}$ , so that  $|\rightarrow^{P_{\text{sp}}, P_{\text{sim}}}| \leq |\rightarrow|$ .

**Contributions.** We present here a novel space and time Efficient Simulation algorithm, called ESim, which features a time complexity in  $O(|P_{\text{sim}}||\rightarrow| \log |\Sigma|)$  and bit space complexity in  $O(|P_{\text{sim}}|^2 \log |P_{\text{sim}}| + |\Sigma| \log |\Sigma|)$ . Thus, ESim reaches the best space bound of GPP and significantly improves the GPP time bound  $O(|P_{\text{sim}}|^2|\rightarrow|)$  by replacing a multiplicative factor  $|P_{\text{sim}}|$  with  $\log |\Sigma|$ . Furthermore, ESim significantly improves the RT space bound  $O(|P_{\text{sim}}||\Sigma| \log |\Sigma|)$  by replacing a multiplicative factor  $|\Sigma| \log |\Sigma|$  with  $|P_{\text{sim}}| \log |P_{\text{sim}}|$  and closely approaches the best time bound of RT.

Analogously to GPP and RT, ESim is a partition refinement algorithm, meaning that it maintains and iteratively refines a so-called partition-relation pair  $\langle P, \preceq \rangle$ , where  $P$  is a partition of the state space  $\Sigma$  that overapproximates the final simulation partition  $P_{\text{sim}}$ , while  $\preceq$  is a binary relation over the partition  $P$  which overapproximates the final simulation preorder. ESim relies on the following three main points, which in particular allow to attain the above complexity bounds.

- (1) Two distinct notions of partition and relation stability for a partition-relation pair are introduced. Accordingly, at high-level, ESim is designed as a partition refinement algorithm which iteratively performs two clearly distinct refinement steps: the refinement of the current partition  $P$  which splits some blocks of  $P$  and the refinement of the relation  $\preceq$  which removes some pairs of blocks from  $P$ .
- (2) ESim exploits an efficient characterization of partition refiners, i.e., blocks of  $P$  that allow to split the current partition  $P$ .
- (3) ESim only relies on data structures, like lists and matrices, that are indexed on and contain blocks of the current partition  $P$ . The hard task here is to devise efficient ways to update these partition-based data structures along the iterations of ESim. We show how this can be actually done, in particular by resorting to Hopcroft’s “process the smaller half” principle [17] when updating a crucial data structure after a partition splitting.

## 2 Background

### 2.1 Notation

If  $R \subseteq \Sigma \times \Sigma$  is any relation and  $X \subseteq \Sigma$  then  $R(X) \triangleq \{x' \in \Sigma \mid \exists x \in X. (x, x') \in R\}$ . Recall that  $R$  is a preorder relation when it is reflexive and transitive. If  $f$  is a function defined on  $\wp(\Sigma)$  and  $x \in \Sigma$  then we often write  $f(x)$  to mean  $f(\{x\})$ . A partition  $P$  of a set  $\Sigma$  is a set of nonempty subsets of  $\Sigma$ , called blocks, that are pairwise disjoint and whose union gives  $\Sigma$ .  $\text{Part}(\Sigma)$  denotes the set of partitions of  $\Sigma$ . If  $P \in \text{Part}(\Sigma)$ ,  $s \in \Sigma$  and  $S \subseteq \Sigma$  then  $P(s)$  denotes the block of  $P$  that contains  $s$  while  $P(S) = \cup_{s \in S} P(s)$ .  $\text{Part}(\Sigma)$  is endowed with the standard partial order  $\preceq$ :  $P_1 \preceq P_2$ , i.e.  $P_2$  is coarser than  $P_1$ , iff for any  $s \in \Sigma$ ,  $P_1(s) \subseteq P_2(s)$ . If  $P_1 \preceq P_2$  and  $B \in P_1$  then  $P_2(B)$  is a block of  $P_2$  which is also denoted by  $\text{parent}_{P_2}(B)$  (when clear from the context the subscript  $P_2$  may be omitted). For a

given nonempty subset  $S \subseteq \Sigma$  called splitter, we denote by  $Split(P, S)$  the partition obtained from  $P$  by replacing each block  $B \in P$  with  $B \cap S$  and  $B \setminus S$  when these sets are nonempty, where we also allow no splitting, namely  $Split(P, S) = P$  (this happens exactly when  $P(S) = S$ ).

A transition system  $(\Sigma, \rightarrow)$  consists of a set  $\Sigma$  of states and of a transition relation  $\rightarrow \subseteq \Sigma \times \Sigma$ . The predecessor/successor transformers  $\text{pre}, \text{post} : \wp(\Sigma) \rightarrow \wp(\Sigma)$  are defined as usual:  $\text{pre}(Y) \triangleq \{s \in \Sigma \mid \exists t \in Y. s \rightarrow t\}$  and  $\text{post}(X) \triangleq \{t \in \Sigma \mid \exists s \in X. s \rightarrow t\}$ . If  $S_1, S_2 \subseteq \Sigma$  then  $S_1 \rightarrow^\exists S_2$  iff there exist  $s_1 \in S_1$  and  $s_2 \in S_2$  such that  $s_1 \rightarrow s_2$ . If  $P \in \text{Part}(\Sigma)$  and  $C \in P$  then  $\text{pre}^\exists(C) \triangleq \{B \in P \mid B \rightarrow^\exists C\}$ . Given a set  $AP$  of atomic propositions (of some specification language), a Kripke structure (KS)  $\mathcal{K} = (\Sigma, \rightarrow, \ell)$  over  $AP$  consists of a transition system  $(\Sigma, \rightarrow)$  together with a state labeling function  $\ell : \Sigma \rightarrow \wp(AP)$ .  $P_\ell \in \text{Part}(\Sigma)$  denotes the state partition induced by  $\ell$ , namely,  $P_\ell \triangleq \{\{s' \in \Sigma \mid \ell(s) = \ell(s')\}\}_{s \in \Sigma}$ .

## 2.2 Simulation Preorder and Equivalence

Let  $\mathcal{K} = (\Sigma, \rightarrow, \ell)$  be a KS. Recall that a relation  $R \subseteq \Sigma \times \Sigma$  is a simulation on  $\mathcal{K}$  if for any  $s, s' \in \Sigma$ , if  $s' \in R(s)$  then: (a)  $\ell(s) = \ell(s')$ ; (b) For any  $t \in \Sigma$  such that  $s \rightarrow t$ , there exists  $t' \in \Sigma$  such that  $s' \rightarrow t'$  and  $t' \in R(t)$ .

Given  $s, t \in \Sigma$ ,  $t$  simulates  $s$ , denoted by  $s \leq t$ , if there exists a simulation relation  $R$  such that  $t \in R(s)$ . The empty relation is a simulation and simulation relations are closed under arbitrary unions so that the largest simulation relation exists. It turns out that the largest simulation on  $\mathcal{K}$  is a preorder relation called simulation preorder and denoted by  $R_{\text{sim}}$ . Thus, for any  $s, t \in \Sigma$ ,  $s \leq t$  iff  $(s, t) \in R_{\text{sim}}$ . Simulation equivalence  $R_{\text{simeq}}$  is the symmetric reduction of  $R_{\text{sim}}$ , namely  $R_{\text{simeq}} \triangleq R_{\text{sim}} \cap R_{\text{sim}}^{-1}$ , so that  $(s, t) \in R_{\text{simeq}}$  iff  $s \leq t$  and  $t \leq s$ .  $P_{\text{sim}} \in \text{Part}(\Sigma)$  denotes the partition corresponding to the equivalence  $R_{\text{simeq}}$  and is called the simulation partition.

## 3 Basic Simulation Algorithm

### 3.1 Partition-Relation Pairs

A *partition-relation pair*  $\mathcal{P} = \langle P, \trianglelefteq \rangle$ , PR for short, is a state partition  $P \in \text{Part}(\Sigma)$  together with a binary relation  $\trianglelefteq \subseteq P \times P$  between blocks of  $P$ . We write  $B \triangleleft C$  when  $B \trianglelefteq C$  and  $B \neq C$  and  $(B', C') \trianglelefteq (B, C)$  when  $B' \trianglelefteq B$  and  $C' \trianglelefteq C$ . When  $\trianglelefteq$  is a preorder/partial order relation then  $\mathcal{P}$  is called, respectively, a preorder/partial order PR.

PRs allow to represent symbolically, i.e. through state partitions, a relation between states. A relation  $R \subseteq \Sigma \times \Sigma$  induces a PR  $\text{PR}(R) = \langle P, \trianglelefteq \rangle$  defined as follows:

- for any  $s$ ,  $P(s) \triangleq \{t \in \Sigma \mid R(s) = R(t)\}$ ;
- for any  $s, t \in \Sigma$ ,  $P(s) \trianglelefteq P(t)$  iff  $t \in R(s)$ .

It is easy to note that if  $R$  is a preorder then  $\text{PR}(R)$  is a partial order PR. On the other hand, a given PR  $\mathcal{P} = \langle P, \trianglelefteq \rangle$  induces the following relation  $\text{Rel}(\mathcal{P})$ :

$$(s, t) \in \text{Rel}(\mathcal{P}) \Leftrightarrow P(s) \trianglelefteq P(t).$$

Here, if  $\mathcal{P}$  is a preorder PR then  $\text{Rel}(\mathcal{P})$  is clearly a preorder.

$\mathcal{P} = \langle P, \trianglelefteq \rangle$  is defined to be a simulation PR on a KS  $\mathcal{K}$  when  $\text{Rel}(\mathcal{P})$  is a simulation on  $\mathcal{K}$ , namely when  $\mathcal{P}$  represents a simulation relation between states. Hence, if  $\mathcal{P}$  is a simulation PR and  $P(s) = P(t)$  then  $s$  and  $t$  are simulation equivalent, while if  $P(s) \trianglelefteq P(t)$  then  $t$  simulates  $s$ .

Given a PR  $\mathcal{P} = \langle P, \trianglelefteq \rangle$ , let us define a map  $\mu_{\mathcal{P}} : \wp(\Sigma) \rightarrow \wp(\Sigma)$  as follows: for any  $X \in \wp(\Sigma)$ ,

$$\mu_{\mathcal{P}}(X) \triangleq \text{Rel}(\mathcal{P})(X) = \cup\{C \in P \mid \exists s \in X. P(s) \trianglelefteq C\}.$$

Note that, for any  $s \in \Sigma$ ,  $\mu_{\mathcal{P}}(s) = \mu_{\mathcal{P}}(P(s)) = \cup\{C \in P \mid P(s) \trianglelefteq C\}$ . This map allows us to characterize for preorder PRs the property of being a simulation as follows.

**Theorem 3.1.** *Let  $\mathcal{P} = \langle P, \trianglelefteq \rangle$  be a preorder PR. Then,  $\mathcal{P}$  is a simulation iff*

- (i) if  $B \sqsubseteq C$ ,  $b \in B$  and  $c \in C$  then  $\ell(b) = \ell(c)$ ;
- (ii) if  $B \rightarrow^{\exists} C$  and  $B \sqsubseteq D$  then  $D \rightarrow^{\exists} \mu_{\mathcal{P}}(C)$ ;
- (iii) for any  $C \in P$ ,  $P = \text{Split}(P, \text{pre}(\mu_{\mathcal{P}}(C)))$ .

*Proof.* ( $\Rightarrow$ ) Condition (i) clearly holds. Assume that  $B \rightarrow^{\exists} C$  and  $B \sqsubseteq D$ . Hence, there exist  $b \in B$  and  $c \in C$  such that  $b \rightarrow c$ . Consider any state  $d \in D$ . Since  $\mathcal{P}$  is a simulation and  $P(b) \sqsubseteq P(d)$ , there exist some state  $e$  such that  $d \rightarrow e$  and  $C = P(c) \sqsubseteq P(e)$ . Hence,  $D \rightarrow^{\exists} \mu_{\mathcal{P}}(C)$ . Finally, if  $C \in P$  and  $x \in \text{pre}(\mu_{\mathcal{P}}(C))$  then there exists some block  $D \sqsupseteq C$  and state  $d \in D$  such that  $x \rightarrow d$ . If  $y \in P(x)$  then since  $P(x) = P(y)$ , by reflexivity of  $\sqsubseteq$ , we have that  $P(x) \sqsubseteq P(y)$ , so that, since  $\mathcal{P}$  is a simulation, there exists some state  $e$  such that  $y \rightarrow e$  and  $P(d) \sqsubseteq P(e)$ . Since  $\sqsubseteq$  is transitive, we have that  $C \sqsubseteq P(e)$ . Hence,  $y \in \text{pre}(\mu_{\mathcal{P}}(C))$ . We have thus shown that  $P(x) \subseteq \text{pre}(\mu_{\mathcal{P}}(C))$ , so that  $P = \text{Split}(P, \text{pre}(\mu_{\mathcal{P}}(C)))$ .

( $\Leftarrow$ ) Let us show that  $\text{Rel}(\mathcal{P})$  is a simulation, i.e., if  $P(s) \sqsubseteq P(s')$  then: (a)  $\ell(s) = \ell(s')$ ; (b) if  $s \rightarrow t$  then there exists  $t'$  such that  $s' \rightarrow t'$  and  $P(t) \sqsubseteq P(t')$ . Condition (a) holds by hypothesis (i). If  $s \rightarrow t$  then  $P(s) \rightarrow^{\exists} P(t)$  so that, by condition (ii), we have that  $P(s') \rightarrow^{\exists} \mu_{\mathcal{P}}(P(t))$ , namely there exists  $s'' \in P(s')$  such that  $s'' \in \text{pre}(\mu_{\mathcal{P}}(P(t)))$ . By condition (iii),  $P(s'') \subseteq \text{pre}(\mu_{\mathcal{P}}(P(t)))$ , i.e.,  $s' \in \text{pre}(\mu_{\mathcal{P}}(P(t)))$ . Hence, there exists  $t'$  such that  $s' \rightarrow t'$  and  $P(t) \sqsubseteq P(t')$ .  $\square$

### 3.2 Partition and Relation Refiners

By Theorem 3.1, assuming that condition (i) holds, there are two possible reasons for a PR  $\mathcal{P} = \langle P, \sqsubseteq \rangle$  for not being a simulation:

1. There exist  $B, C, D \in P$  such that  $B \rightarrow^{\exists} C$ ,  $B \sqsubseteq D$ , but  $D \not\rightarrow^{\exists} \mu_{\mathcal{P}}(C)$ ; in this case we say that the block  $C$  is a *relation refiner* for  $\mathcal{P}$ .
2. There exist  $B, C \in P$  such that  $B \cap \text{pre}(\mu_{\mathcal{P}}(C)) \neq \emptyset$  but  $B \not\subseteq \text{pre}(\mu_{\mathcal{P}}(C))$ ; in this case we say that the block  $C$  is a *partition refiner* for  $\mathcal{P}$ .

We therefore define

$$\text{RRefiner}(\mathcal{P}) \triangleq \{C \in P \mid C \text{ is a relation refiner for } \mathcal{P}\}$$

$$\text{PRefiner}(\mathcal{P}) \triangleq \{C \in P \mid C \text{ is a partition refiner for } \mathcal{P}\}$$

and we say that  $\mathcal{P}$  is relation or partition *stable* when, respectively,  $\text{RRefiner}(\mathcal{P}) = \emptyset$  or  $\text{PRefiner}(\mathcal{P}) = \emptyset$ . Then, Theorem 3.1 reads as follows:  $\mathcal{P}$  is a simulation iff  $\mathcal{P}$  satisfies condition (i) and is both relation and partition stable.

If  $C \in \text{PRefiner}(\mathcal{P})$  then  $P$  is first refined to  $P' \triangleq \text{Split}(P, \text{pre}(\mu_{\mathcal{P}}(C)))$ , i.e.  $P$  is split w.r.t. the splitter  $S = \text{pre}(\mu_{\mathcal{P}}(C))$ . Accordingly, the relation  $\sqsubseteq$  can be transformed into the following relation  $\sqsubseteq'$  defined on  $P'$ :

$$\sqsubseteq' \triangleq \{(D, E) \in P' \times P' \mid \text{parent}_P(D) \sqsubseteq \text{parent}_P(E)\} \quad (1)$$

Two blocks  $D$  and  $E$  of the refined partition  $P'$  are related by  $\sqsubseteq'$  if their parent blocks  $\text{parent}_P(D)$  and  $\text{parent}_P(E)$  in  $P$  were related by  $\sqsubseteq$ . Hence, if  $\mathcal{P}' = \langle P, \sqsubseteq' \rangle$  then for all  $D \in P'$ , we have that  $\mu_{\mathcal{P}'}(D) = \mu_{\mathcal{P}}(\text{parent}_P(D))$ . We will show that this refinement of  $\langle P, \sqsubseteq \rangle$  is correct because if  $B \in P$  is split into  $B \setminus S$  and  $B \cap S$  then all the states in  $B \setminus S$  are not simulation equivalent to all the states in  $B \cap S$ . Note that if  $B \in P$  has been split into  $B \cap S$  and  $B \setminus S$  then both  $B \cap S \sqsubseteq' B \setminus S$  and  $B \setminus S \sqsubseteq' B \cap S$  hold, and consequently  $\mathcal{P}'$  becomes relation unstable.

On the other hand, if  $\mathcal{P}$  is partition stable and  $C \in \text{RRefiner}(\mathcal{P})$  then we will show that  $\sqsubseteq$  can be safely refined to the following relation  $\sqsubseteq'$ :

$$\begin{aligned} \sqsubseteq' \triangleq \sqsubseteq \setminus \{(B, D) \in P \times P \mid B \rightarrow^{\exists} C, B \sqsubseteq D, D \not\rightarrow^{\exists} \mu_{\mathcal{P}}(C)\} \\ = \{(B, D) \in P \times P \mid B \sqsubseteq D, (B \rightarrow^{\exists} C \Rightarrow D \rightarrow^{\exists} \mu_{\mathcal{P}}(C))\} \end{aligned} \quad (2)$$

because if  $(B, D) \in \sqsubseteq \setminus \sqsubseteq'$  then all the states in  $D$  cannot simulate all the states in  $B$ .

The above ideas lead us to design a basic simulation algorithm ESIm described in Figure 1. ESIm maintains a PR  $\mathcal{P} = \langle P, \sqsubseteq \rangle$  which is initialized as  $\langle P_{\ell}, \text{id} \rangle$  and then iteratively refined as follows:

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1 ESim(PR  $\langle P, \sqsubseteq \rangle$ ) {
2   Initialize(); PStabilize(); bool PStable := RStabilize(); bool RStable := tt;
3   while  $\neg(\text{PStable} \ \& \ \text{RStable})$  do
4     if  $\neg \text{PStable}$  then {RStable := PStabilize(); PStable := tt}
5     if  $\neg \text{RStable}$  then {PStable := RStabilize(); RStable := tt}
6   }
7   bool PStabilize() {
8      $P_{\text{old}} := P$ ;
9     while  $\exists C \in \text{PRefiner}(\mathcal{P})$  do
10       $S := \text{pre}(\mu_{\mathcal{P}}(C))$ ;  $P := \text{Split}(S)$ ;
11      forall  $(D, E) \in P \times P$  do  $D \sqsubseteq E := \text{parent}(D) \sqsubseteq \text{parent}(E)$ ;
12    return  $(P = P_{\text{old}})$ ;
13  }
14  bool RStabilize() {
15    // Precondition: PStable is tt
16     $\sqsubseteq_{\text{old}} := \sqsubseteq$ ; Delete :=  $\emptyset$ ;
17    while  $\exists C \in \text{RRefiner}(\mathcal{P})$  do
18      Delete := Delete  $\cup \{(B, D) \in P \times P \mid B \sqsubseteq D, B \rightarrow^{\exists} C, D \not\rightarrow^{\exists} \mu_{\mathcal{P}}(C)\}$ ;
19     $\sqsubseteq := \sqsubseteq \setminus \text{Delete}$ ;
20    return  $(\sqsubseteq = \sqsubseteq_{\text{old}})$ ;
21  }

```

Figure 1: Basic Simulation Algorithm.

- (A) If  $\langle P, \sqsubseteq \rangle$  is not partition stable then the partition  $P$  is split for  $\text{pre}(\mu_{\mathcal{P}}(C))$  as long as a partition refiner  $C$  for  $\mathcal{P}$  exists, and when this happens the relation  $\sqsubseteq$  is transformed to  $\sqsubseteq'$  as defined by (1); at the end of this process, we have a PR  $\mathcal{P}' = \langle P', \sqsubseteq' \rangle$  which is partition stable and if  $P$  has been actually refined, i.e.  $P' \prec P$  then the current PR  $\mathcal{P}'$  becomes relation unstable.
- (B) If  $\langle P, \sqsubseteq \rangle$  is not relation stable then the relation  $\sqsubseteq$  is refined to  $\sqsubseteq'$  as described by (2) as long as a relation refiner for  $\mathcal{P}$  exists; hence, at the end of this refinement process  $\langle P, \sqsubseteq' \rangle$  becomes relation stable but possibly partition unstable.

Moreover, the following properties of the current PR of ESim hold.

**Lemma 3.2.** *In any run of ESim, the following two conditions hold:*

- (i) *If PStabilize is called on a partial order PR  $\langle P, \sqsubseteq \rangle$  then at the exit we obtain a PR  $\langle P', \sqsubseteq' \rangle$  which is a preorder.*
- (ii) *If RStabilize is called on a preorder PR  $\langle P, \sqsubseteq \rangle$  then at the exit we obtain a PR  $\langle P, \sqsubseteq' \rangle$  which is a partial order.*

*Proof.* Let us first consider *PStabilize*. Consider an input partial order PR  $\mathcal{P} = \langle P, \sqsubseteq \rangle$ , a splitter  $S$  such that  $P' = \text{Split}(P, S)$  and let  $\sqsubseteq'$  be defined as in equation (1). Let us show that  $\langle P', \sqsubseteq' \rangle$  is a preorder PR.

(Reflexivity): If  $B \in P'$  then, as  $\sqsubseteq$  is reflexive,  $P(B) \sqsubseteq P(B)$  and thus  $B \sqsubseteq' B$ .

(Transitivity): Assume that  $B, C, D \in P'$  and  $B \sqsubseteq' C$  and  $C \sqsubseteq' D$ . Then,  $P(B) \sqsubseteq P(C)$  and  $P(C) \sqsubseteq P(D)$ , so that by transitivity of  $\sqsubseteq$ ,  $P(B) \sqsubseteq P(D)$ . Hence,  $B \sqsubseteq' D$ .

Let us then take into account *RStabilize*, consider an input preorder PR  $\mathcal{P} = \langle P, \sqsubseteq \rangle$  and let  $\langle P, \sqsubseteq' \rangle$  be the output PR of *RStabilize*.

(Reflexivity): If  $B \in P$  then, by reflexivity of  $\sqsubseteq$ ,  $B \sqsubseteq B$ . If  $B \rightarrow^{\exists} C$ , for some  $C \in P$ , then since  $C \sqsubseteq C$  by reflexivity of  $\sqsubseteq$ , we have that  $B \rightarrow^{\exists} \mu_{\mathcal{P}}(C)$ . Hence,  $B \sqsubseteq' B$ .

(Transitivity): Assume that  $B \trianglelefteq' C$  and  $C \trianglelefteq' D$ . Then,  $B \trianglelefteq C$  and  $C \trianglelefteq D$ , so that by transitivity of  $\trianglelefteq$ ,  $B \trianglelefteq D$ . If  $B \rightarrow^{\exists} E$  then, since  $B \trianglelefteq' C$ ,  $C \rightarrow^{\exists} \mu_{\mathcal{P}}(E)$ . Hence, there exists  $F \in P$  such that  $E \trianglelefteq F$  and  $C \rightarrow^{\exists} F$ . Since  $C \trianglelefteq' D$ , we have that  $D \rightarrow^{\exists} \mu_{\mathcal{P}}(F)$ . Since  $\trianglelefteq$  is transitive and  $E \trianglelefteq F$ ,  $\mu_{\mathcal{P}}(F) \subseteq \mu_{\mathcal{P}}(E)$ . Thus, we have shown that  $B \rightarrow^{\exists} E$  implies  $D \rightarrow^{\exists} \mu_{\mathcal{P}}(E)$ , namely  $B \trianglelefteq' D$ .

(Antisymmetry): We observe that after calling *PStabilize* on a partial order PR, antisymmetry can be lost because for any block  $B$  which is split into  $B_1 = B \cap S$  and  $B_2 = B \setminus S$ , where  $S = \text{pre}(\mu_{\mathcal{P}}(C))$ , we have that  $B_1 \trianglelefteq B_2$  and  $B_2 \trianglelefteq B_1$ . In this case, *RStabilize* removes the pair  $(B \cap S, B \setminus S)$  from the relation  $\trianglelefteq$ : in fact, while  $B \cap S \subseteq \text{pre}(\mu_{\mathcal{P}}(C))$  and therefore  $B \cap S \rightarrow^{\exists} E$ , for some block  $E \subseteq \mu_{\mathcal{P}}(C)$ , we have that  $(B \setminus S) \cap \text{pre}(\mu_{\mathcal{P}}(C)) = \emptyset$ , so that, since  $\mu_{\mathcal{P}}(E) \subseteq \mu_{\mathcal{P}}(C)$ ,  $(B \setminus S) \cap \text{pre}(\mu_{\mathcal{P}}(E)) = \emptyset$ , i.e.,  $B \setminus S \not\rightarrow^{\exists} \mu_{\mathcal{P}}(E)$ , and therefore  $B \cap S \not\trianglelefteq' B \setminus S$ . Hence, *RStabilize* outputs a relation  $\trianglelefteq'$  which is antisymmetric.  $\square$

The main loop of ESim terminates when the current PR  $\langle P, \trianglelefteq \rangle$  becomes both partition and relation stable. By the Lemma 3.2, the output PR  $\mathcal{P}$  of ESim is a partial order, and hence a preorder, so that Theorem 3.1 can be applied to  $\mathcal{P}$  which is then a simulation PR. It turns out that this algorithm is correct, meaning that the output PR  $\mathcal{P}$  represents the simulation preorder.

**Theorem 3.3 (Correctness).** *Let  $\Sigma$  be finite. ESim is correct, i.e., if  $\langle P, \trianglelefteq \rangle$  is the output PR of ESim on input  $\langle P_{\ell}, \text{id} \rangle$  then for any  $s, t \in \Sigma$ ,  $s \leq t \Leftrightarrow P(s) \trianglelefteq P(t)$ .*

*Proof.* Let us first note that ESim always terminates. In fact, if  $\langle P, \trianglelefteq \rangle$  is the current PR at the beginning of some iteration of the while-loop of ESim and  $\langle P', \trianglelefteq' \rangle$  is the current PR at the beginning of the next iteration then, since  $\langle P', \trianglelefteq' \rangle$  is either partition or relation unstable, we have that either  $P' \triangleleft P$  or  $P' = P$  and  $\trianglelefteq' \subsetneq \trianglelefteq$ . Since the state space  $\Sigma$  is finite, at some iteration it must happen that  $P' = P$  and  $\trianglelefteq' = \trianglelefteq$  so that PStable & RStable = tt.

When ESim terminates, we have that  $\text{RRefiner}(\langle P, \trianglelefteq \rangle) = \emptyset = \text{PRefiner}(\langle P, \trianglelefteq \rangle)$ . Also, let us observe that condition (i) of Theorem 3.1 always holds for the current PR  $\langle P, \trianglelefteq \rangle$  because the input PR  $\langle P_{\ell}, \text{id} \rangle$  initially satisfies condition (i) and this condition is clearly preserved at any iteration of ESim. Furthermore, at the beginning, we have that  $\langle P, \trianglelefteq \rangle = \langle P_{\ell}, \text{id} \rangle$  and this is trivially a partial order. Thus, we can apply Lemma 3.2 for any call to *PStabilize* and *RStabilize*, so that we obtain that the output PR  $\langle P, \trianglelefteq \rangle$  is a preorder. Hence, Theorem 3.1 can be applied to the output preorder PR  $\langle P, \trianglelefteq \rangle$ , which is then a simulation. Thus,  $\text{Rel}(\langle P, \trianglelefteq \rangle) \subseteq R_{\text{sim}}$ .

Conversely, let us show that if  $\mathcal{P}$  is the output PR of ESim then  $R_{\text{sim}} \subseteq \text{Rel}(\mathcal{P})$ . This is shown as follows: if  $\mathcal{P}$  is a preorder PR such that  $R_{\text{sim}} \subseteq \text{Rel}(\mathcal{P})$  and *RStabilize* or *PStabilize* are called on  $\mathcal{P}$  then at the exit we obtain a PR  $\mathcal{P}'$  such that  $R_{\text{sim}} \subseteq \text{Rel}(\mathcal{P}')$ .

Let us first take into account *RStabilize*, consider an input preorder PR  $\mathcal{P} = \langle P, \trianglelefteq \rangle$  such that  $R_{\text{sim}} \subseteq \text{Rel}(\mathcal{P})$ , and let  $\mathcal{P}' = \langle P', \trianglelefteq' \rangle$  be the output PR of *RStabilize*. We show that  $R_{\text{sim}} \subseteq \text{Rel}(\mathcal{P}')$ , that is, for any  $s, t \in \Sigma$ , if  $s \leq t$  then  $P(s) \trianglelefteq' P(t)$ . By hypothesis, from  $s \leq t$  we obtain  $P(s) \trianglelefteq P(t)$ . Assume that  $P(s) \rightarrow^{\exists} C$ , for some  $C \in P$ . Hence,  $P(s) \rightarrow^{\exists} \mu_{\mathcal{P}}(C)$ . Since the PR  $\mathcal{P}$  is partition stable, we have that  $P(s) \subseteq \text{pre}(\mu_{\mathcal{P}}(C))$ . Thus, there exists some  $D \in P$  and  $d \in D$  such that  $C \trianglelefteq D$  and  $s \rightarrow d$ . Therefore, since  $t$  simulates  $s$ , there exists some state  $e$  such that  $t \rightarrow e$  and  $d \leq e$ . By hypothesis, from  $d \leq e$  we obtain  $D = P(d) \trianglelefteq P(e)$ . Hence, from  $C \trianglelefteq D$  and  $D \trianglelefteq P(e)$ , since  $\trianglelefteq$  is transitive, we obtain  $C \trianglelefteq P(e)$ . Thus,  $t \rightarrow^{\exists} \mu_{\mathcal{P}}(C)$  and in turn  $P(t) \rightarrow^{\exists} \mu_{\mathcal{P}}(C)$ . We can thus conclude that  $P(s) \trianglelefteq' P(t)$ .

Let us now consider *PStabilize*. Consider an input preorder PR  $\mathcal{P} = \langle P, \trianglelefteq \rangle$  (which, by Lemma 3.2, actually is a partial order PR) such that  $R_{\text{sim}} \subseteq \text{Rel}(\mathcal{P})$ . Consider a splitter  $S$  such that  $P' = \text{Split}(P, S)$  and let  $\trianglelefteq'$  be defined as in equation (1). Let  $\mathcal{P}' = \langle P', \trianglelefteq' \rangle$  and let us check that  $R_{\text{sim}} \subseteq \text{Rel}(\mathcal{P}')$ , i.e., if  $s \leq t$  then  $P'(s) \trianglelefteq' P'(t)$ . By hypothesis, if  $s \leq t$  then  $P(s) \trianglelefteq P(t)$ . Moreover, by definition of  $\trianglelefteq'$  and since  $P' \preceq P$ ,  $P(s) \trianglelefteq P(t)$  iff  $P'(s) \trianglelefteq' P'(t)$ .

To sum up, we have shown that for the output PR  $\langle P, \trianglelefteq \rangle$ ,  $R_{\text{sim}} = \text{Rel}(\langle P, \trianglelefteq \rangle)$ , so that  $s \leq t$  iff  $P(s) \trianglelefteq P(t)$ .  $\square$

```

1 Initialize() {
2   // Initialize BCount
3   forall B ∈ P do
4     [ forall C ∈ P do BCount(B, C) := 0;
5   forall B ∈ P do
6     [ forall x ∈ B do
7       [ forall y ∈ post(x) do
8         [ if (BCount(B, y.block) = 0) then BCount(B, y.block) := 1;
9   // Initialize preE
10  updatePreE(); // In Figure 5
11  // Initialize Count
12  forall B ∈ P do
13    [ forall C ∈ P do Count(B, C) := 0;
14  forall D ∈ P do
15    [ forall B ∈ D.preE do
16      [ forall C ∈ P such that C ≤ D do Count(B, C)++;
17  // Initialize Rem
18  forall C ∈ P do
19    [ forall B ∈ P do
20      [ forall D ∈ B.preE do
21        [ if (Count(D, C) = 0) then Rem(C).append(D);
22 }

```

Figure 2: Initialization of data structures.

## 4 Efficient Implementation

### 4.1 Data Structures

ESim is implemented by using the following data structures.

**States:** A state  $s$  is represented by a record that contains the list  $\text{post}(s)$  of its successors, a pointer  $s.\text{block}$  to the block  $P(s)$  that contains  $s$  and a boolean flag used for marking. The state space  $\Sigma$  is represented as a doubly linked list of states.  $\{\text{post}(s)\}_{s \in \Sigma}$  therefore represents the input transition system.

**Partition:** The states of any block  $B$  of the current partition  $P$  are consecutive in the list  $\Sigma$ , so that  $B$  is represented by two pointers  $\text{begin}$  and  $\text{end}$ :  $B.\text{begin}$  is the first state of  $B$  in  $\Sigma$  and  $B.\text{end}$  is the successor of the last state of  $B$  in  $\Sigma$ , i.e.,  $B = [B.\text{begin}, B.\text{end}]$ . Moreover,  $B$  stores a boolean flag  $B.\text{intersection}$  and a block pointer  $B.\text{brother}$  whose meanings are as follows: after a call to  $\text{Split}(P, S)$  for splitting  $P$  w.r.t. a set of states  $S$ , if  $B_1 = B \cap S$  and  $B_2 = B \setminus S$ , for some  $B \in P$  that has been split by  $S$  then  $B_1.\text{intersection} = \mathbf{tt}$  and  $B_2.\text{intersection} = \mathbf{ff}$ , while  $B_1.\text{brother}$  points to  $B_2$  and  $B_2.\text{brother}$  points to  $B_1$ . If instead  $B$  has not been split by  $S$  then  $B.\text{intersection} = \mathbf{null}$  and  $B.\text{brother} = \mathbf{null}$ . Also, any block  $B$  stores in  $\text{Rem}(B)$  a list of blocks of  $P$ , which is used by  $R\text{Stabilize}$ , and in  $B.\text{preE}$  the list of blocks  $C \in P$  such that  $C \rightarrow^3 B$ . Finally, any block  $B$  stores in  $B.\text{size}$  the size of  $B$ , in  $B.\text{count}$  an integer counter bounded by  $|P|$  which is used by  $P\text{Stabilize}$  and a pair of boolean flags used for marking. The current partition  $P$  is stored as a doubly linked list of blocks.

**Relation:** The current relation  $\leq$  on  $P$  is stored as a resizable  $|P| \times |P|$  boolean matrix. Recall [8, Section 17.4] that insert operations in a resizable array (whose capacity is doubled as needed) take amortized constant time, and that a resizable matrix (or table) can be implemented as a resizable array of resizable arrays. The boolean matrix  $\leq$  is resized by adding a new entry to  $\leq$ , namely a new row and a new column, for any block  $B$  that is split into two new blocks  $B \setminus S$  and  $B \cap S$ . The old entry  $B$  becomes the entry for the new block  $B \setminus S$  while the new entries represent the new block  $B \cap S$ .

```

1 bool PStabilize() {
2   list(Block) split := ∅;
3   while (C := FindPRefiner()) ≠ null do
4     list(State) S := preμ(C); split := Split(S);
5     updateRel(split); updateBCount(split); updatePreE();
6     updateCount(split); updateRem(split);
7   return (split = ∅);
8 }

9 Block FindPRefiner() {
10  forall B ∈ P do
11    list(Block) p := Post(B);
12    forall C ∈ p do
13      if (Count(B, C) = 1) then return C;
14  return null;
15 }

16 list(Block) Post(Block B) {
17  list(Block) p := ∅;
18  forall b ∈ B do
19    forall c ∈ post(b) do
20      Block C := c.block;
21      if unmarked1(C) then {mark1(C); C.count = 0; p.append(C); }
22      if unmarked2(C) then {mark2(C); C.count++;}
23    forall C ∈ p do unmark2(C);
24  forall C ∈ p do
25    unmark1(C);
26    if (C.count = B.size) then p.remove(C);
27  return p;
28 }

```

Figure 3: PStabilize Algorithm.

**Auxiliary Data Structures:** We store and maintain a resizable boolean matrix BCount and a resizable integer matrix Count, both indexed over  $P$  and having the following meanings:

$$\text{BCount}(B, C) \triangleq \begin{cases} 1 & \text{if } B \rightarrow^{\exists} C \\ 0 & \text{if } B \not\rightarrow^{\exists} C \end{cases}$$

$$\text{Count}(B, C) \triangleq \sum_{E \supseteq C} \text{BCount}(B, E).$$

Hence,  $\text{Count}(B, C)$  stores the number of blocks  $E$  such that  $C \leq E$  and  $B \rightarrow^{\exists} E$ . The table Count allows to implement the test  $B \not\rightarrow^{\exists} \text{pre}(\mu_{\mathcal{P}}(C))$  in constant time as  $\text{Count}(B, C) = 0$ .

The data structures BCount, preE, Count and Rem are initialized by the function *Initialize* at line 2 of ESim, which is described in Figure 2.

## 4.2 Partition Stability

Our efficient implementation of ESim will rely on the following characterization of partition refiners.

**Theorem 4.1.** *Let  $\langle P, \trianglelefteq \rangle$  be a partial order PR. Then,  $\text{PRefiner}(\langle P, \trianglelefteq \rangle) \neq \emptyset$  iff there exists  $B, C \in P$  such that the following three conditions hold:*

- (i)  $B \rightarrow^{\exists} C$ ;
- (ii) for any  $C' \in P$ , if  $C \triangleleft C'$  then  $B \not\rightarrow^{\exists} C'$ ;
- (iii)  $B \not\leq \text{pre}(C)$ .



```

1 list(State) preμ(Block C) {
2   list(State) S := ∅;
3   forall x ∈ Σ do
4     forall y ∈ post(x) do
5       if (C ≤ y.block & unmarked(x)) then {S.append(x); mark(x);}
6   forall x ∈ S do unmark(x);
7   return S;
8 }

9 list(Block) Split(list(State) S) {
10  list(Block) split; forall B ∈ P do B.intersection := null;
11  forall x ∈ S do
12    if (x.block.intersection = null) then
13      x.block.intersection := ff;
14      Block B := new Block; x.block.brother := B;
15      B.brother := x.block; B.intersection := tt;
16      split.append(x.block);
17  move x in the list Σ from x.block at the end of B;
18  if (x.block = ∅) then // x.block ⊆ S
19    x.block.begin := B.begin; x.block.end := B.end;
20    x.block.brother := null; x.block.intersection := null;
21    split.remove(x.block); delete B;
22  return split;
23 }

```

Figure 4: *Split Algorithm*.

*Proof.* Let  $\mathcal{P} = \langle P, \leq \rangle$ .

( $\Leftarrow$ ) From condition (i) we have that  $B \cap \text{pre}(\mu_{\mathcal{P}}(C)) \neq \emptyset$ . From conditions (ii) and (iii),  $B \not\subseteq \text{pre}(\mu_{\mathcal{P}}(C))$ . Thus,  $C \in \text{PRefiner}(\mathcal{P})$ .

( $\Rightarrow$ ) Assume that  $\text{PRefiner}(\mathcal{P}) \neq \emptyset$ . Since  $\langle P, \leq \rangle$  is a partial order, we consider a partition refiner  $C \in \max(\text{PRefiner}(\mathcal{P}))$  which is maximal w.r.t. the partial order  $\leq$ . Since  $C$  is a partition refiner, there exists some  $B \in P$  such that  $B \cap \text{pre}(\mu_{\mathcal{P}}(C)) \neq \emptyset$  and  $B \not\subseteq \text{pre}(\mu_{\mathcal{P}}(C))$ . If  $C' \in P$  is such that  $C \triangleleft C'$  then  $C'$  cannot be a partition refiner because  $C$  is a maximal partition refiner. Hence, if  $B \rightarrow^{\exists} C'$  then  $B \subseteq \text{pre}(\mu_{\mathcal{P}}(C'))$ , because  $C'$  is not a partition refiner, so that, since  $\text{pre}(\mu_{\mathcal{P}}(C')) \subseteq \text{pre}(\mu_{\mathcal{P}}(C))$ ,  $B \subseteq \text{pre}(\mu_{\mathcal{P}}(C))$ , which is a contradiction. Hence, for any  $C' \in P$  if  $C \triangleleft C'$  then  $B \not\rightarrow^{\exists} C'$ . Therefore, from  $B \cap \text{pre}(\mu_{\mathcal{P}}(C)) \neq \emptyset$  we obtain that  $B \rightarrow^{\exists} C$ . Moreover, from  $B \not\subseteq \text{pre}(\mu_{\mathcal{P}}(C))$  we obtain that  $B \not\subseteq \text{pre}(C)$ .  $\square$

Notice that this characterization of partition refiners requires that the current PR is a partial order relation and, by Lemma 3.2, for any call to *PStabilize*, this is actually guaranteed by the *ESim* algorithm.

The algorithm in Figure 3 is an implementation of the basic algorithm *PStabilize* that relies on Theorem 4.1 and on the above data structures. The function *FindPRefiner* implements the conditions of Theorem 4.1: it returns a partition refiner for the current PR  $\mathcal{P} = \langle P, \leq \rangle$  when this exists otherwise returns a null pointer. Given a block  $B \in P$ , the function call *Post*( $B$ ) returns a list of blocks  $C \in P$  that satisfy conditions (i) and (iii) of Theorem 4.1, i.e., those blocks  $C$  such that  $B \rightarrow^{\exists} C$  and  $B \not\subseteq \text{pre}(C)$ . This is accomplished through the counter  $C.\text{count}$  that at the exit of the for-loop at lines 18-23 stores the number of states in  $B$  having (at least) an outgoing transitions to  $C$ , i.e.,  $C.\text{count} = |B \cap \text{pre}(C)|$ . Hence, we have that:

$$B \rightarrow^{\exists} C \text{ and } B \not\subseteq \text{pre}(C) \Leftrightarrow 1 \leq C.\text{count} < B.\text{size}.$$

Then, for any candidate partition refiner  $C \in \text{Post}(B)$ , it remains to check condition (ii) of Theorem 4.1. This condition is checked in *FindPRefiner* by testing whether  $\text{Count}(B, C) = 1$ : this is correct because  $\text{Count}(B, C) \geq 1$  holds since  $C \in \text{Post}(B)$  and therefore  $B \rightarrow^{\exists} C$ , so that

$$\text{Count}(B, C) = 1 \text{ iff } \forall C' \in P. C \triangleleft C' \Rightarrow B \not\rightarrow^{\exists} C'.$$

```

1  updateRel(list(Block) split) {
2    forall B ∈ split do addNewEntry(B) in matrix ≤;
3    forall B ∈ P do
4      forall C ∈ split do
5        if (B.intersection = tt) then B ≤ C := B.brother ≤ C.brother;
6        else B ≤ C := B ≤ C.brother;
7    forall C ∈ P do
8      forall B ∈ split do
9        if (C.intersection = ff) then B ≤ C := B.brother ≤ C;
10 }

11 updateBCount(list(Block) split) {
12   forall B ∈ split do addNewEntry(B) in matrix Count;
13   forall B ∈ P do
14     forall x ∈ B do
15       forall y ∈ post(x) do BCount(B, y.block) := 0;
16   forall B ∈ P do
17     forall x ∈ B do
18       forall y ∈ post(x) do
19         if (BCount(B, y.block) = 0) then BCount(B, y.block) := 1;
20 }

21 updatePreE() {
22   forall B ∈ P do B.preE := ∅;
23   forall B ∈ P do
24     forall x ∈ B do
25       forall y ∈ post(x) do {unmark(B); y.block.preE.append(B);}
26   forall C ∈ P do
27     forall B ∈ C.preE do
28       if unmarked(B) then mark(B);
29       else C.preE.remove(B);
30   forall B ∈ C.preE do unmark(B);
31 }

32 updateRem(list(Block) split) {
33   forall B ∈ split do Rem(B) := Rem(B.brother);
34 }

```

Figure 5: *update* functions.

Hence, if  $\text{Count}(B, C) = 1$  holds at line 13 of *FindPRefiner*, by Theorem 4.1,  $C$  is a partition refiner. Once a partition refiner  $C$  has been returned by *Post(B)*, *PStabilize* splits the current partition  $P$  w.r.t. the splitter  $S = \text{pre}(\mu_{\mathcal{P}}(C))$  by calling the function *Split(S)*, updates the relation  $\leq$  as defined by equation 1 in Section 3.2 by calling *updateRel*, updates the data structures BCount, preE, Count and Rem, and then check again whether a partition refiner exists. At the exit of the main while-loop of *PStabilize*, the current PR  $\langle P, \leq \rangle$  is partition stable.

*PStabilize* calls the functions *preμ* and *Split* that are described in Figure 4. Recall that the states of a block  $B$  of  $P$  are consecutive in the list of states  $\Sigma$ , so that  $B$  is represented as  $B = [B.\text{begin}, B.\text{end}]$ . The implementation of *Split(S)* is quite standard (see e.g. [13, 21]) and only scans the states in  $S$ , i.e. *Split(S)* takes  $O(|S|)$  time. This operation affects the ordering of the states in the list  $\Sigma$  because states are moved from old blocks to newly generated blocks.

We will show that the overall time complexity of *PStabilize* along a run of ESim is in  $O(|P_{\text{sim}}| \rightarrow | \rangle)$ .

### 4.3 Updating Data Structures

In the function *PStabilize*, after calling *Split(S)*, firstly we need to update the boolean matrix that stores the relation  $\sqsubseteq$  in accordance with definition (1) in Section 3.2. After that, since both  $P$  and  $\sqsubseteq$  are changed we need to update the data structures *BCount*, *preE*, *Count* and *Rem*. In Figure 5 we describe the implementations of the functions *updateRel*, *updateBCount*, *updatePreE* and *updateRem*, which are quite simple and easy to understand.

The function *updateCount* is in Figure 6 and deserves special care in order to design a time efficient implementation. The core part of the *updateCount* algorithm follows Hopcroft’s “process the smaller half” principle [17] for updating the integer matrix *Count*. Let  $P'$  be the partition which is obtained by splitting the partition  $P$  w.r.t. the splitter  $S$ . Let  $B$  be a block of  $P$  that has been split into  $B \cap S$  and  $B \setminus S$ . Thus, we need to update  $\text{Count}(B \cap S, C)$  and  $\text{Count}(B \setminus S, C)$  for any  $C \in P'$  by knowing  $\text{Count}(B, \text{parent}_P(C))$ . Let us first observe that after lines 3-10 of *updateCount*, we have that for any  $B, C \in P'$ ,  $\text{Count}(B, C) = \text{Count}(\text{parent}_P(B), \text{parent}_P(C))$ . Let  $X$  be the block in  $\{B \cap S, B \setminus S\}$  with the smaller size, and let  $Z$  be the other block, so that  $|X| \leq |Z|$  and  $|X| + |Z| = |B|$ . Let  $C$  be any block in  $P'$ . We set  $\text{Count}(X, C)$  to 0, while  $\text{Count}(Z, C)$  is left unchanged, namely  $\text{Count}(Z, C) = \text{Count}(B, C)$ . We can correctly update both  $\text{Count}(Z, C)$  and  $\text{Count}(X, C)$  by just scanning all the outgoing transitions from  $X$ . In fact, if  $x \in X$ ,  $x \rightarrow y$  and the block  $P(y)$  is scanned for the first time then for all  $C \sqsubseteq P(y)$ ,  $\text{Count}(X, C)$  is incremented by 1 while if  $Z \not\rightarrow^3 P(y)$ , i.e.  $\text{BCount}(Z, P(y)) = 0$ , then  $\text{Count}(Z, C)$  is decremented by 1. The correctness of this procedure goes as follows:

- (1) At the end,  $\text{Count}(X, C)$  is clearly correct because its value has been re-computed from scratch;
- (2) At the end,  $\text{Count}(Z, C)$  is correct because  $\text{Count}(Z, C)$  initially stores the value  $\text{Count}(B, C)$ , and if there exists some block  $D$  such that  $C \sqsubseteq D$ ,  $B \rightarrow^3 D$  whereas  $Z \not\rightarrow^3 D$  — this is correctly implemented at line 26 as  $\text{BCount}(Z, D) = 0$ , since the data structure *BCount* is up to date — then necessarily  $X \rightarrow^3 D$ , because  $B$  has been split into  $X$  and  $Z$ , so that  $D = P(y)$  for some  $y \in \text{post}(X)$ , namely  $D$  has been taken into account by some increment  $\text{Count}(X, C)++$  at line 25 and consequently  $\text{Count}(Z, C)$  is decremented by 1 at line 26.

Moreover, if some block  $D \in P' \setminus \{B \cap S, B \setminus S\}$  is such that both  $D \rightarrow^3 X$  and  $D \rightarrow^3 Z$  hold then for all the blocks  $C \in P$  such that  $C \sqsubseteq X$  (or, equivalently,  $C \sqsubseteq Z$ ), we need to increment  $\text{Count}(D, C)$  by 1. This is done at lines 28-30 by relying on the updated data structures *preE* and *BCount*.

Let us observe that the time complexity of a single call of *updateCount(split)* is

$$|P| \left( |split| + \sum_{X \in split} (|\{(x, y) \mid x \in X, y \in \Sigma, x \rightarrow y\}| + |\{(X, D) \mid D \in P, X \rightarrow^3 D\}|) \right).$$

Hence, let us calculate the overall time complexity of *updateCount*. If  $X$  and  $X'$  are two blocks that are scanned in two different calls of *updateCount* and  $X' \subseteq X$  then  $|X'| \leq |X|/2$ . Consequently, any transition  $x \rightarrow y$  at line 21 and  $D \rightarrow^3 X$  at line 28 can be scanned in some call of *updateCount* at most  $\log_2 |\Sigma|$  times. Thus, the overall time complexity of *updateCount* is in  $O(|P_{\text{sim}}| |\rightarrow| \log |\Sigma|)$ .

### 4.4 Relation Stability

The basic procedure *RStabilize* in Figure 1 is implemented by the algorithm in Figure 7. Let  $\mathcal{P}^{\text{in}} = \langle P, \sqsubseteq^{\text{in}} \rangle$  be the current PR when calling *RStabilize*. For each relation refiner  $C \in P$ , *RStabilize* must iteratively refine the initial relation  $\sqsubseteq^{\text{in}}$  in accordance with equation (2) in Section 3.2. Hence, if  $B \rightarrow^3 C$ ,  $B \sqsubseteq D$  and  $D \not\rightarrow^3 \mu_{\mathcal{P}^{\text{in}}}(C)$ , the entry  $B \sqsubseteq D$  of the boolean matrix that represents the relation  $\sqsubseteq$  must be set to **ff**. Thus, the idea is to store and incrementally maintain for each block  $C \in P$  a list *Rem*( $C$ ) of blocks  $D \in P$  such that:

- (A) If  $C$  is a relation refiner for  $\mathcal{P}^{\text{in}}$  then  $\text{Rem}(C) \neq \emptyset$ ;
- (B) If  $D \in \text{Rem}(C)$  then necessarily  $D \not\rightarrow^3 \mu_{\mathcal{P}^{\text{in}}}(C)$ .

```

1 // Precondition: BCount and preE are updated with the current PR
2 updateCount(list(Block) split) {
3   forall B ∈ split do addNewEntry(B) in matrix Count;
4   forall B ∈ P do
5     forall C ∈ split do
6       if (B.intersection = tt) then Count(B, C) := Count(B.brother, C.brother);
7       else Count(B, C) := Count(B, C.brother);
8   forall C ∈ P do
9     forall B ∈ split do
10      if (C.intersection = ff) then Count(B, C) := Count(B.brother, C);
11  forall C ∈ P do unmark(C);
12  forall B ∈ split do
13    // Update Count(B, ·) and Count(B.brother, ·)
14    Block X, Z;
15    if (B.size ≤ B.brother.size) then
16      {X := B; Z := B.brother;}
17    else
18      {X := B.brother; Z := B;}
19    forall C ∈ P do {Count(X, C) := 0; /* Count(Z, C) := Count(B, C); */}
20    forall x ∈ X do
21      forall y ∈ post(x) do
22        if unmarked(y.block) then
23          mark(y.block);
24          forall C ∈ P such that C ≤ y.block do
25            Count(X, C)++;
26            if (BCount(Z, y.block) = 0) then Count(Z, C)--;
27    // For all D ∉ {B, B.brother}, update Count(D, ·)
28    forall D ∈ X.preE do
29      if (D ≠ X & D ≠ Z & BCount(D, Z) = 1) then
30        forall C ∈ P such that C ≤ X do Count(D, C)++;
31  }

```

Figure 6: *updateCount* function.

It turns out that  $C$  is a relation refiner for  $\mathcal{P}^{\text{in}}$  iff there exist blocks  $B$  and  $D$  such that  $B \rightarrow^{\exists} C$ ,  $D \in \text{Rem}(C)$  and  $B \leq D$ . Hence, the set of blocks  $\text{Rem}(C)$  is reminiscent of the set of states  $\text{remove}(s)$  used in Henzinger et al.'s [15] simulation algorithm, since each pair  $(B, D)$  which must be removed from the relation  $\leq$  is such that  $D \in \text{Rem}(C)$ , for some block  $C$ .

Initially, namely at the first call of *RStabilize* by *ESim*,  $\text{Rem}(C)$  is set by the function *Initialize* in Figure 2 to  $\{D \in P \mid D \rightarrow^{\exists} \Sigma, D \not\rightarrow^{\exists} \mu_{\mathcal{P}}(C)\}$ . Hence, *RStabilize* scans all the blocks in the current partition  $P$  and selects those blocks such that  $\text{Rem}(C) \neq \emptyset$ , which are therefore candidate to be relation refiners. Then, by scanning all the blocks  $B \in C.\text{preE}$  and  $D \in \text{Rem}(C)$ , if  $B \leq D$  holds then the entry  $B \leq D$  must be set to **ff**. However, the removal of the pair  $(B, D)$  from the current relation  $\leq$  may affect the function  $\mu_{\mathcal{P}}$ . This is avoided by making a copy  $\text{oldRem}(C)$  of all the  $\text{Rem}(C)$ 's at the beginning of *RStabilize* and then using this copy. During the main for-loop of *RStabilize*,  $\text{Rem}(C)$  must therefore satisfies the following invariant property:

$$(\text{Inv}): \forall C \in P. \text{Rem}(C) = \{D \in P \mid D \rightarrow^{\exists} \mu_{\mathcal{P}}^{\text{in}}(C), D \not\rightarrow^{\exists} \mu_{\mathcal{P}}(C)\}.$$

This means that at the beginning of *RStabilize*, any  $\text{Rem}(C)$  is set to empty, and after the removal of a pair  $(B, D)$  from  $\leq$ , since  $\mu_{\mathcal{P}}(B)$  changes, we need: (i) to update the matrix *Count*, for all the entries  $(F, B)$  where  $F \rightarrow^{\exists} D$ , and (ii) to check if there is some block  $F$  such that  $F \not\rightarrow^{\exists} \mu_{\mathcal{P}}(B)$ , because any such  $F$  must be added to  $\text{Rem}(B)$  in order to maintain the invariant property (Inv).

```

1 bool RStabilize() {
2   //  $\mu_P^{\text{in}} := \mu_P$ ;
3   forall  $C \in P$  do {oldRem(C) := Rem(C); Rem(C) =  $\emptyset$ ; }
4   bool Removed := ff;
5   forall  $C \in P$  such that oldRem(C)  $\neq \emptyset$  do
6     // Invariant (Inv):  $\forall C \in P. \text{Rem}(C) = \{D \in P \mid D \rightarrow^{\exists} \mu_P^{\text{in}}(C), D \not\rightarrow^{\exists} \mu_P(C)\}$ 
7     forall  $B \in C.\text{preE}$  do
8       forall  $D \in \text{oldRem}(C)$  do
9         if ( $B \preceq D$ ) then
10            $B \preceq D := \text{ff}$ ; Removed := tt;
11           // update Count and Rem
12           forall  $F \in D.\text{preE}$  do
13             Count(F, B) := Count(F, B) - 1;
14             if (Count(F, B) = 0) then
15               if (Rem(B) =  $\emptyset$ ) then //  $F \rightarrow^{\exists} \mu_P^{\text{in}}(B) \ \& \ F \not\rightarrow^{\exists} \mu_P(B)$ 
16                 Rem(B).append(F);
17   return  $\neg \text{Removed}$ ;
18 }

```

Figure 7: RStabilize Algorithm.

## 5 Complexity

The time complexity bound of the algorithm ESim relies on the following key properties:

- (1) The overall number of partition refiners found by ESim is bounded by  $|P_{\text{sim}}| - |P_\ell|$  and therefore is in  $O(|P_{\text{sim}}|)$ . Moreover, the overall number of newly generated blocks by the splitting operations performed by calling *Split(S)* at line 5 of *PStabilize* is in  $O(|P_{\text{sim}}|)$ . In fact, let  $\{P_i\}_{i \in [0, n]}$  be the sequence of different partitions computed by ESim where  $P_0$  is the initial partition  $P_\ell$ ,  $P_n$  is the final partition  $P_{\text{sim}}$  and for all  $i \in [1, n]$ ,  $P_i$  is the partition after the  $i$ -th call to *Split(S)*, so that  $P_i \prec P_{i-1}$ . The number of newly generated blocks by a call *Split(S)* refines  $P_i$  to  $P_{i+1}$  is  $2(|P_{i+1}| - |P_i|)$ . Thus, the overall number of newly generated blocks is  $\sum_{i=1}^n 2(|P_i| - |P_{i-1}|) = 2(|P_{\text{sim}}| - |P_\ell|) \in O(|P_{\text{sim}}|)$ .
- (2) The invariant (Inv) of the sets  $\text{Rem}(C)$  guarantees the following property: if  $C_1$  and  $C_2$  are two blocks that are selected by the for-loop at line 5 of *RStabilize* in two different calls of *RStabilize*, and  $C_2 \subseteq C_1$  (possibly  $C_1 = C_2$ ) then  $(\cup \text{Rem}(C_1)) \cap (\cup \text{Rem}(C_2)) = \emptyset$ .

**Theorem 5.1.** ESim runs in  $O(|P_{\text{sim}}|^2 \log |P_{\text{sim}}| + |\Sigma| \log |\Sigma|)$ -space and  $O(|P_{\text{sim}}| |\rightarrow| \log |\Sigma|)$ -time.

*Proof. Space Complexity.* The input transition system is represented by the post relation, so that the size of post is not taken into account in the space complexity of ESim. The doubly linked list of states take  $O(|\Sigma| \log |\Sigma|)$  while the pointers *s.block* take  $O(|\Sigma| \log |P_{\text{sim}}|)$ . The partition  $P$  and the pointers stored in each block of  $P$  overall take  $O(|P_{\text{sim}}| \log |\Sigma|)$ . The binary relation  $\preceq$  takes  $O(|P_{\text{sim}}|^2)$ . The auxiliary data structures *Rem*, *preE* and *BCount* overall take  $O(|P_{\text{sim}}|^2)$ , while the integer matrix *Count* takes  $O(|P_{\text{sim}}|^2 \log |P_{\text{sim}}|)$ . Hence, the overall bit space complexity for storing the above data structures is  $O(|P_{\text{sim}}|^2 \log |P_{\text{sim}}| + |\Sigma| \log |\Sigma|)$ .

**Time Complexity.** The time complexity bound of ESim is shown by the following points.

- (A) The initialization function *Initialize* takes  $|P|^2 + |\rightarrow| + |P| |\{(B, D) \mid B, D \in P, B \rightarrow^{\exists} D\}|$  time. Observe that  $|P| \leq |P_{\text{sim}}| \leq |\rightarrow|$  so that the time complexity of *Initialize* is in  $O(|P_{\text{sim}}| |\rightarrow|)$ .
- (B) A call to  $\text{pre}\mu(C)$  takes  $O(|\rightarrow|)$  time. A call to *Split(S)* takes  $|S|$  time. Since  $S$  is returned by  $\text{pre}\mu(C)$ ,  $|S| \leq |\rightarrow|$  holds so that the time complexity of a call to *Split(S)* is in  $O(|\rightarrow|)$ . A call to *Post(B)* takes  $|\{(b, c) \mid b \in B, c \in \Sigma, b \rightarrow c\}|$  time, so that a call to *FindPRefiner* takes  $O(|\rightarrow|)$  time.

Moreover, let us observe that *FindPRefiner* returns **null** just once, because when *FindPRefiner* returns **null** the current PR of ESim is both partition and relation stable and therefore ESim terminates and outputs that PR. Consequently, since, by point (1) above, the overall number of partition refiners is in  $O(|P_{\text{sim}}|)$ , the overall number of function calls for *FindPRefiner* is in  $O(|P_{\text{sim}}|)$  and, in turn, the overall time complexity of *FindPRefiner* is in  $O(|P_{\text{sim}}||\rightarrow|)$  time. Also, the overall time complexity of  $\text{pre}\mu(C)$  and *Split*( $S$ ) is in  $O(|P_{\text{sim}}||\rightarrow|)$ .

(C) Let us observe that the calls *updateRel*(*split*) and *updateRem*(*split*) take  $O(|P||\text{split}|)$  time, while *updatePreE*() and *updateBCount*(*split*) take  $O(|\rightarrow|)$  time. Since the overall number of calls for these functions is in  $O(|P_{\text{sim}}|)$  and since  $\sum_{i \in \text{Iterations}} |\text{split}_i|$  is in  $O(|P_{\text{sim}}|)$ , it turns out that their overall time complexity is in  $O(|P_{\text{sim}}|(|P_{\text{sim}}| + |\rightarrow|))$ , so that, since  $|P_{\text{sim}}| \leq |\rightarrow|$ , it is in  $O(|P_{\text{sim}}||\rightarrow|)$ . Moreover, as already shown in Section 4.3, the overall time complexity of *updateCount*(*split*) is in  $O(|P_{\text{sim}}||\rightarrow| \log |\Sigma|)$ .

(D) Hence, by points (B) and (C), the overall time complexity of *PStabilize* is in  $O(|P_{\text{sim}}||\rightarrow| \log |\Sigma|)$ .

(E) Let  $C \in P_{\text{in}}$  be some block of the initial partition and let  $\langle C_i \rangle_{i \in I_C}$ , for some set of indices  $I_C$ , be a sequence of blocks selected by the for-loop at line 5 of *RStabilize* such that: (a) for any  $i \in I_C$ ,  $C_i \subseteq C$  and (b) for any  $i$ ,  $C_{i+1}$  has been selected after  $C_i$  and  $C_{i+1}$  is contained in  $C_i$ . Observe that  $C$  is the parent block in  $P_{\text{in}}$  of all the  $C_i$ 's. Then, by the property (2) above, it turns out that the corresponding sets in  $\{\cup \text{Rem}(C_i)\}_{i \in I_C}$  are pairwise disjoint so that  $\sum_{i \in I_C} |\text{Rem}(C_i)| \leq |P_{\text{sim}}|$ . This property guarantees that if  $D \in \text{oldRem}(C_i)$  at line 8 then for all the blocks  $D' \subseteq D$  and for any  $j \in I_C$  such that  $i < j$ ,  $D' \notin \text{oldRem}(C_j)$ . Moreover, if the test  $B \trianglelefteq D$  at line 9 is true for some iteration  $k$ , so that  $B \trianglelefteq D$  is set to **ff**, then for all the blocks  $D'$  and  $B'$  such that  $D' \subseteq D$  and  $B' \subseteq B$  the test  $D' \trianglelefteq B'$  will be always false for all the iterations which follow  $k$ . From these observations, we derive that the overall time complexity of the code of the for-loop at lines 7-10 is  $\sum_C \sum_{i \in I_C} \sum_{B \rightarrow \exists C} |\text{Rem}(C_i)| \leq |P_{\text{sim}}| |\{(B, C) \mid B, C \in P_{\text{sim}}, B \rightarrow \exists C\}| \leq |P_{\text{sim}}||\rightarrow|$ . Moreover, the overall time complexity of the code of the for-loop at lines 12-16 is  $\sum_B \sum_D \sum_{F \rightarrow \exists D} 1 \leq |P_{\text{sim}}| |\{(F, D) \mid F, D \in P_{\text{sim}}, F \rightarrow \exists D\}| \leq |P_{\text{sim}}||\rightarrow|$ . We also observe that the overall time complexity of the for-loop at line 3 of *RStabilize* is in  $O(|P_{\text{sim}}|^2)$ . Thus, the overall time complexity of *RStabilize* is in  $O(|P_{\text{sim}}|(|P_{\text{sim}}| + |\rightarrow|))$ , so that, since  $|P_{\text{sim}}| \leq |\rightarrow|$ , it is in  $O(|P_{\text{sim}}||\rightarrow|)$ .

Summing up, by points (A), (D) and (E), we have shown that the overall time complexity of ESim is in  $O(|P_{\text{sim}}||\rightarrow| \log |\Sigma|)$ .  $\square$

## 6 Conclusion and Further Work

We have introduced a new algorithm, called ESim, for efficiently computing the simulation preorder which: (i) reaches the space bound of the simulation algorithm GPP [10, 11] featuring the best space complexity, while significantly improving its time bound; (ii) significantly improves the space bound of the simulation algorithm RT [20, 21] featuring the best time complexity, while closely approaching its time bound. Moreover, the space complexity of ESim is quasi-optimal, meaning that it differs only for logarithmic factors from the size of the output.

We see a couple of interesting avenues for further work. A first natural question arises: can the time complexity of ESim be further improved and reaches the time complexity of RT? This would require to eliminate the multiplicative factor  $\log |\Sigma|$  from the time complexity of ESim and, at present, this seems quite hard to achieve. More in general, it would be interesting to investigate whether some lower space and time bounds can be stated for the simulation preorder problem. Secondly, ESim works for Kripke structures. While an adaptation of a simulation algorithm from Kripke structures to labeled transition systems (LTSs) can be conceptually simple, unfortunately such a shift may lead to some loss in both space and time complexities. We mention [1, 16] and [18] that provide simulation algorithms for LTSs by adapting, respectively, RT and GPP. Thus, it is worth investigating how ESim can be efficiently adapted to work with LTSs.

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